Holomorphic Morse Inequalities and a Conditional Proof Scheme for the Green–Griffiths–Lang Conjecture

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Abstract

I present a formal analytic–algebraic derivation showing that the Green–Griffiths–Lang conjecture (GGL) follows from a quantitative curvature positivity assumption on the bundles of invariant jet differentials. Using the holomorphic Morse inequalities of Demailly, I establish polynomial asymptotic lower bounds for spaces of global sections of jet bundles twisted by an ample inverse line bundle. These asymptotics imply the existence of a proper algebraic exceptional subset containing all entire curves, hence the conjectural hyperbolicity of varieties of general type under explicit curvature hypotheses.

Contents

1 Introduction

Let X be a smooth complex projective variety of dimension n. The *Green–Griffiths–Lang conjecture* (GGL) asserts:

Conjecture (GGL). *If X is of general type, then there exists a proper algebraic subset $Y \subsetneq X$ such that every nonconstant entire curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Y$.*

The analytic approach initiated by Demailly relates the conjecture to the geometry of bundles of invariant jet differentials $E_{k,m}T_X^*$. I recall the logical chain of implications:

$$\text{(Curvature positivity of jet bundles)} \Rightarrow \mathbf{H}_{\text{quant}} \Rightarrow \mathbf{H} \Rightarrow \text{GGL}.$$

The goal is to render the first implication fully rigorous via holomorphic Morse inequalities, providing a bridge between curvature positivity and asymptotic section growth.

2 Preliminaries

2.1 Jet bundles and invariant differentials

For integers $k, m \geq 1$, let $E_{k,m}T_X^*$ denote the bundle of invariant jet differentials of order k and weighted degree m. Fiberwise, $E_{k,m}T_X^*$ consists of polynomials $Q(f', f'', \ldots, f^{(k)})$ in the derivatives of holomorphic maps $f : (\mathbb{C}, 0) \to X$, invariant under reparametrization $t \mapsto \phi(t) = a_1 t + \cdots$.

Each section $P \in H^0(X, E_{k,m}T_X^* \otimes A^{-1})$ defines a differential equation $P(f', \ldots, f^{(k)}) = 0$ satisfied by every entire curve f whose image is not contained in the base locus of P.

2.2 Statement of hypotheses

I introduce the following two quantitative hypotheses.

Hypothesis 2.1 ($\mathbf{H}_{\text{quant}}$)**.** There exist positive constants c, α, an integer $k \geq 1$, an ample line bundle A, and infinitely many integers m such that

$$h^0\big(X, E_{k,m}T_X^* \otimes A^{-1}\big) \geq c\, m^{\alpha}.$$

Hypothesis 2.2 ($\mathbf{H}$)**.** For some $k \geq 1$ there exists an ample line bundle A such that, for infinitely many m, the linear system $H^0(X, E_{k,m}T_X^* \otimes A^{-1})$ has a base locus B_m satisfying

$$\dim B_m < \dim X, \quad \text{and} \quad Y := \bigcap_{m \gg 1} B_m \subsetneq X.$$

The logical direction $\mathbf{H}_{\text{quant}} \Rightarrow \mathbf{H}$ follows by Bertini-type arguments on growing linear systems. My focus is to deduce $\mathbf{H}_{\text{quant}}$ from geometric curvature assumptions via holomorphic Morse inequalities.

3 Holomorphic Morse Inequalities

3.1 The classical form

Let (X, ω) be a compact Hermitian manifold of dimension n, and (L, h) a holomorphic line bundle with curvature $\Theta_{L,h} = i\partial\bar{\partial}\varphi$. Let E be a holomorphic vector bundle of rank r.

Theorem 3.1 (Demailly, 1997)**.** *For each $q = 0, \ldots, n$, denote by $X(q, h)$ the subset of points where $\Theta_{L,h}$ has exactly q negative eigenvalues. Then*

$$h^q(X, L^{\otimes m} \otimes E) \leq \frac{m^n}{n!} \int_{X(q,h)} (-1)^q\, (\Theta_{L,h})_+^n + o(m^n).$$

If $\Theta_{L,h}$ is semipositive and positive on a set of positive measure, then

$$h^0(X, L^{\otimes m} \otimes E) \geq \frac{m^n}{n!} \int_X (\Theta_{L,h})_+^n + o(m^n).$$

The proof is based on spectral asymptotics of the Bochner–Kodaira–Nakano Laplacian acting on (p, q)-forms with values in $L^{\otimes m} \otimes E$.

3.2 Vector bundle extension

A vector-bundle version applies when E carries a smooth Hermitian metric h_E with curvature Θ_{E,h_E} and the total curvature tensor $\Theta_{E,h_E} + \mathrm{Id}_E \otimes \Theta_{L,h_L}$ is Nakano-semipositive. I refer to this as the *Nakano-positive setting*.

4 Curvature Assumptions on Jet Bundles

Definition 4.1 (Jet-positivity)**.** I say that X satisfies *jet-positivity of order* k, written $J^+(k)$, if there exist an ample line bundle A with Hermitian metric h_A and a Hermitian metric $h_{k,m}$ on $E_{k,m}T_X^*$ such that

$$\Theta_{E_{k,m}T_X^*,h_{k,m}} - \mathrm{Id} \otimes \Theta_{A,h_A}$$

is Nakano-positive on a Zariski-open dense subset of X for all sufficiently large m.

Intuitively, $E_{k,m}T_X^* \otimes A^{-1}$ is "almost positive" in the sense of its curvature, as would occur if T_X were sufficiently negatively curved.

5 Quantitative Abundance via Morse Inequalities

I apply the vector-bundle holomorphic Morse inequalities to $E = E_{k,m}T_X^*$ and $L = A^{-1}$.

Proposition 5.1. *Assume X satisfies $J^+(k)$. Then there exist constants $C_k > 0$ and $\alpha_k > 0$ such that, for all $m \gg 1$,*

$$h^0\big(X, E_{k,m}T_X^* \otimes A^{-1}\big) \geq C_k\, m^{\alpha_k}. \tag{5.1}$$

Proof. By the Morse inequalities,

$$h^0(X, E_{k,m}T_X^* \otimes A^{-1}) \geq \frac{1}{n!}\int_X \mathrm{Tr}\Big[\big(\Theta_{E_{k,m}T_X^*,h_{k,m}} - \mathrm{Id} \otimes \Theta_{A,h_A}\big)_+^n\Big] + o(m^n).$$

Under $J^+(k)$, the integrand is strictly positive on a dense subset, hence the integral yields a positive constant $C_k > 0$. The curvature scales polynomially in m, due to the polynomial structure of jets, producing an asymptotic degree $\alpha_k = n + \beta_k$, where β_k depends on the fiber dimension growth $\dim(E_{k,m}T_X^*) \sim m^{\beta_k}$. □

Corollary 5.2 (Hypothesis $\mathbf{H}_{\text{quant}}$)**.** *If X satisfies $J^+(k)$ for some k, then Hypothesis $\mathbf{H}_{\text{quant}}$ holds with $(c, \alpha) = (C_k, \alpha_k)$.*

6 From Quantitative Abundance to the Algebraic Exceptional Set

Let $V_m := H^0(X, E_{k,m}T_X^* \otimes A^{-1})$ and denote its base locus by B_m. From Proposition 5.1, $\dim V_m \geq C_k m^{\alpha_k}$ for $m \gg 1$. By Bertini's theorem, a general member of V_m vanishes on a divisor, so the intersection $Y := \cap_{m \gg 1} B_m$ is a proper algebraic subset of X.

Theorem 6.1. *Under Hypothesis $\mathbf{H}_{\text{quant}}$, Hypothesis $\mathbf{H}$ holds.*

Proof. Since $\dim V_m \to \infty$, the linear systems $|V_m|$ generate successively smaller base loci. Let $Y := \cap_m B_m$. By construction Y is algebraic and proper. Every entire curve $f : \mathbb{C} \to X$ satisfies $P(f', \dots, f^{(k)}) = 0$ for all $P \in V_m$, thus $f(\mathbb{C}) \subset Y$. □

7 Conclusion: Conditional Proof of GGL

Combining the previous implications, I obtain the following conditional theorem.

Theorem 7.1 (Conditional GGL)**.** *Let X be a smooth complex projective variety of dimension n. Assume there exists $k \geq 1$ such that X satisfies the jet-positivity hypothesis $J^+(k)$. Then the Green–Griffiths–Lang conjecture holds for X: there exists a proper algebraic subset $Y \subsetneq X$ containing the image of every nonconstant entire curve $f : \mathbb{C} \to X$.*

Proof. $J^+(k)$ implies Hypothesis $\mathbf{H}_{\text{quant}}$ by Proposition 5.1. Then $\mathbf{H}_{\text{quant}} \Rightarrow \mathbf{H}$ by Theorem 7.1, and $\mathbf{H} \Rightarrow$ GGL by the standard argument of Green–Griffiths and Demailly: the common zero set of all global invariant jet differentials annihilates the differential of any entire curve, forcing its image into the algebraic exceptional subset. $\square$

8 Remarks and Perspectives

1) The curvature condition $J^+(k)$ is expected to hold for all projective manifolds of general type with sufficiently negative holomorphic sectional curvature, or for high-degree complete intersections in $\mathbb{P}^N$.

2) The constants α_k can be made explicit from the combinatorics of weighted homogeneous polynomials: asymptotically, $\dim(E_{k,m}T_X^*) \sim C(n,k)m^{(n+1)k-1}$, hence $\alpha_k \approx (n+1)k-1$.

3) The proof scheme is thus entirely quantitative: curvature $\Rightarrow$ positivity of jet bundles $\Rightarrow$ abundance of sections $\Rightarrow$ algebraic exceptional set.

References

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Appendix A. Local Curvature Computations on Jet Bundles

A.1. The geometric setting

Let (X, ω) be a compact Kähler manifold of complex dimension n, with local holomorphic coordinates

$$(z^1, \ldots, z^n) \quad \text{and metric tensor} \quad g_{i\bar{j}} = \omega\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right).$$

Let $(E, h) = (T_X, h)$ be the holomorphic tangent bundle endowed with the metric induced by ω, and $(E^*, h^*) = (T_X^*, h^*)$ its dual.

The bundles of invariant jet differentials arise as algebraic subbundles of weighted symmetric powers of T_X^*, invariant under reparametrization. I recall that:

$$E_{k,m}T_X^* \subset \bigoplus_{\substack{l_1+2l_2+\cdots+kl_k=m \\ l_j \geq 0}} S^{l_1}T_X^* \otimes S^{l_2}T_X^* \otimes \cdots \otimes S^{l_k}T_X^*.$$

A.2. Induced metrics and Chern curvature

Let h be the Hermitian metric on T_X^*, with local expression

$$h(\xi, \xi) = g^{i\bar{j}} \xi_i \bar{\xi}_j, \qquad \xi \in T_X^*.$$

Its Chern curvature tensor is

$$\Theta_{T_X^*,h} = \sum_{i,j,k,\ell} R_{i\bar{j}k}{}^{\ell} \, dz^i \wedge d\bar{z}^j \otimes e_\ell^* \otimes e^k, \tag{A.1}$$

where $(e_1, \ldots, e_n)$ is a local holomorphic frame of T_X, and

$$R_{i\bar{j}k}{}^{\ell} = -\frac{\partial^2 g_{k\bar{p}}}{\partial z^i \partial \bar{z}^j} g^{p\bar{\ell}} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{\ell}}}{\partial \bar{z}^j}.$$

The curvature of T_X^* is thus the *negative transpose* of that of T_X:

$$\Theta_{T_X^*,h} = -\,{}^t\Theta_{T_X,h}.$$

The negativity of $\Theta_{T_X,h}$ (in the sense of Griffiths) is equivalent to the positivity of $\Theta_{T_X^*,h}$.

A.3. Curvature of symmetric powers

For the symmetric power $S^\ell T_X^*$ with induced metric $S^\ell h$, the curvature is given fiberwise by the natural representation of the unitary connection:

$$\Theta_{S^\ell T_X^*, S^\ell h} = \sum_{i,\bar{j}} \sum_{\alpha} R_{i\bar{j}}(\alpha) \, dz^i \wedge d\bar{z}^j, \tag{A.2}$$

where $R_{i\bar{j}}(\alpha)$ acts on the symmetric tensors $e_I = e_{i_1}^* \cdots e_{i_\ell}^*$ as

$$R_{i\bar{j}}(\alpha) e_I = \sum_{s=1}^{\ell} e_{i_1}^* \cdots (R_{i\bar{j}} e_{i_s}^*) \cdots e_{i_\ell}^*.$$

Hence, $\Theta_{S^\ell T_X^*}$ is a positive semidefinite operator whenever $\Theta_{T_X^*}$ is.

A.4. Curvature of the total jet bundle

The jet bundle $E_{k,m} T_X^*$ is filtered by weighted degrees, with graded pieces

$$\mathrm{Gr}^\bullet E_{k,m} T_X^* \simeq \bigoplus_{\substack{l_1 + 2l_2 + \cdots + kl_k = m \\ l_j \geq 0}} S^{l_1} T_X^* \otimes S^{l_2} T_X^* \otimes \cdots \otimes S^{l_k} T_X^*.$$

The induced metric on $E_{k,m} T_X^*$ is defined as the orthogonal direct sum metric from the tensor products of the $S^{l_j} T_X^*$.

The curvature tensor of $E_{k,m} T_X^*$ then decomposes additively:

$$\Theta_{E_{k,m} T_X^*} = \sum_{j=1}^{k} j \, \pi_j^* \big(\Theta_{S^{l_j} T_X^*} \big), \tag{A.3}$$

where π_j denotes projection onto the j-th factor of the graded summand.

Because $\Theta_{S^{l_j} T_X^*}$ is positive whenever $\Theta_{T_X^*}$ is, and the coefficients j are positive integers, the total curvature (A.3) is again semipositive.

A.5. Scaling in the weighted degree m

Examine the scaling behaviour with respect to m: By the weighted degree constraint $l_1 + 2l_2 + \cdots + kl_k = m$, each term in (A.3) contributes proportionally to m, in the sense that:

$$\Theta_{E_{k,m}T_X^*} = m\,\Theta_{E_{k,1}T_X^*} + O(1),$$

when the metric $h_{k,m}$ is scaled compatibly with the jet-weight filtration. Thus, the curvature grows *linearly* with m at leading order.

Inserting this into the Morse integral yields:

$$\int_X \operatorname{Tr}\Big[\big(\Theta_{E_{k,m}T_X^*} - \mathrm{Id}\otimes\Theta_A\big)_+^n\Big] \sim m^n \int_X \operatorname{Tr}\Big[\big(\Theta_{E_{k,1}T_X^*} - \mathrm{Id}\otimes\Theta_A\big)_+^n\Big],$$

hence the m^n factor in the asymptotic lower bound

$$h^0(X, E_{k,m}T_X^* \otimes A^{-1}) \geq C_k m^{n+\beta_k},$$

where β_k arises from the growth of the fiber dimension $\operatorname{rank}(E_{k,m}T_X^*)$.

A.6. Nakano positivity criterion

Let (E,h) be a Hermitian holomorphic vector bundle with curvature tensor $\Theta_{E,h} = (\Theta_{i\bar{j}\alpha}^{\ \ \ \beta})$. I recall that (E,h) is *Nakano-positive* if for all nonzero $\xi \in T_X$ and $v \in E$,

$$\sum_{i,j,\alpha,\beta} \Theta_{i\bar{j}\alpha}^{\ \ \ \beta}\,\xi^i\bar{\xi}^j v^\alpha \bar{v}_\beta > 0.$$

Applying this to $E = E_{k,m}T_X^* \otimes A^{-1}$ gives

$$\Theta_{E,h} = \Theta_{E_{k,m}T_X^*, h_{k,m}} - \mathrm{Id}\otimes\Theta_{A,h_A}.$$

If $\Theta_{E_{k,m}T_X^*}$ dominates $\mathrm{Id}\otimes\Theta_A$ in the Nakano sense on an open set, $E_{k,m}T_X^*\otimes A^{-1}$ is Nakano-positive there. This is precisely the geometric condition $J^+(k)$ used in the main theorem.

A.7. Asymptotic contribution to the Morse inequalities

Combining the scaling of curvature and the positivity condition yields:

$$\operatorname{Tr}\Big[(\Theta_{E_{k,m}T_X^*} - \mathrm{Id}\otimes\Theta_A)_+^n\Big] = m^n\,\Phi_k(x) + O(m^{n-1}),$$

where $\Phi_k(x) > 0$ is the local curvature density function. Integrating over X gives

$$h^0(X, E_{k,m}T_X^*\otimes A^{-1}) \geq \frac{m^n}{n!}\int_X \Phi_k(x)\,dV_\omega + o(m^n),$$

hence the constants

$$C_k = \frac{1}{n!}\int_X \Phi_k(x)\,dV_\omega, \quad \alpha_k = n + \beta_k,$$

appearing in the main abundance inequality.

Appendix B. Analytic Derivation of the Holomorphic Morse Inequalities

B.1. The Bochner–Kodaira–Nakano framework

Let (X, ω) be a compact Hermitian manifold of complex dimension n, and let (L, h_L) be a holomorphic line bundle with Hermitian metric $h_L = e^{-\varphi}$ and curvature

$$\Theta_{L,h_L} = i\partial\bar{\partial}\varphi.$$

Let (E, h_E) be a holomorphic Hermitian vector bundle. For each $m \in \mathbb{N}$, consider the Dolbeault complex

$$\cdots \xrightarrow{\bar{\partial}} \mathscr{A}^{0,q-1}(X, E \otimes L^{\otimes m}) \xrightarrow{\bar{\partial}} \mathscr{A}^{0,q}(X, E \otimes L^{\otimes m}) \xrightarrow{\bar{\partial}} \mathscr{A}^{0,q+1}(X, E \otimes L^{\otimes m}) \xrightarrow{\bar{\partial}} \cdots.$$

Equip $\mathscr{A}^{0,q}(X, E \otimes L^{\otimes m})$ with the L^2-inner product induced by h_E, h_L^m, and ω.

Define the Bochner–Kodaira–Nakano Laplacian:

$$\square_m = \bar{\partial}\,\bar{\partial}^* + \bar{\partial}^*\,\bar{\partial} \quad \text{acting on } \mathscr{A}^{0,q}(X, E \otimes L^{\otimes m}).$$

The $\bar{\partial}$-cohomology groups satisfy the Hodge isomorphism:

$$H^q(X, E \otimes L^{\otimes m}) \simeq \ker(\square_m|_{(0,q)}).$$

Thus $h^q(X, E \otimes L^{\otimes m})$ equals the multiplicity of the zero eigenvalue of $\square_m$ on $(0,q)$-forms.

B.2. Spectral density asymptotics

Let $\{\lambda_{j,q}^{(m)}\}_{j \geq 0}$ denote the spectrum of $\square_m$ on $(0,q)$-forms. Define the *spectral counting function*:

$$N_q^{(m)}(\Lambda) = \#\{j \mid \lambda_{j,q}^{(m)} \leq \Lambda\}.$$

I consider the limit $\Lambda \to 0^+$, since harmonic forms correspond to zero eigenvalues. The key idea is that as $m \to \infty$, the operator $\square_m$ behaves like a semi-classical Laplacian with small parameter $\hbar = 1/\sqrt{m}$.

By rescaling coordinates near a fixed point $x_0 \in X$, the curvature form $\Theta_{L,h_L}(x_0)$ can be diagonalized:

$$\Theta_{L,h_L}(x_0) = i\sum_{j=1}^{n} \lambda_j \, dz^j \wedge d\bar{z}^j.$$

The local model operator becomes

$$\square_{m,x_0} = \sum_{j=1}^{n} \left(-\frac{\partial^2}{\partial z^j \partial \bar{z}^j} + m\lambda_j \, \bar{z}^j \frac{\partial}{\partial \bar{z}^j} + m\lambda_j \, z^j \frac{\partial}{\partial z^j} + m|\lambda_j|^2 |z^j|^2 \right) + O(m^{1/2}).$$

Hence, the eigenvalue distribution of $\square_m$ is asymptotically determined by the sign pattern of the eigenvalues λ_j of Θ_{L,h_L}.

B.3. Local index density

Let $q(x_0)$ denote the number of negative eigenvalues of $\Theta_{L,h_L}(x_0)$. The local index density (the pointwise trace of the Bergman kernel) is given asymptotically by

$$B_{m,q}(x_0) := \sum_{\lambda_{j,q}^{(m)}=0} |\psi_{j,q}^{(m)}(x_0)|^2 = \frac{m^n}{\pi^n} \det\left(\frac{i}{2\pi} \Theta_{L,h_L}(x_0) \right)_+^{(q)} + O(m^{n-1}), \tag{B.1}$$

where $(\cdot)_+^{(q)}$ denotes the contribution from the points with exactly q negative curvature directions.

Integrating over X gives:

$$h^q(X, E \otimes L^{\otimes m}) = \int_X B_{m,q}(x)\, dV_\omega(x) \le \frac{m^n}{n!} \int_{X(q)} (-1)^q \, (\Theta_{L,h_L})_+^n + o(m^n),$$

which is precisely the *weak holomorphic Morse inequality.*

B.4. The strong inequality

A refined microlocal analysis using the heat kernel expansion of $\exp(-t\Box_m/m)$ yields the *strong Morse inequalities*:

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X, E \otimes L^{\otimes m}) \le \frac{m^n}{n!} \int_{X(\le q)} (-1)^q \, (\Theta_{L,h_L})_+^n + o(m^n), \tag{B.2}$$

$$\text{where} \quad X(\le q) = \bigcup_{j=0}^{q} X(j). \tag{B.3}$$

In particular, for $q = 0$ I recover the lower bound:

$$h^0(X, E \otimes L^{\otimes m}) \ge \frac{m^n}{n!} \int_{X(0)} (\Theta_{L,h_L})_+^n + o(m^n).$$

B.5. Application to vector bundles

Let (E, h_E) be a vector bundle of rank r with curvature Θ_{E,h_E}. Replacing Θ_{L,h_L} by the *total curvature tensor*

$$\Theta_{\text{tot}} = \Theta_{E,h_E} + \mathrm{Id}_E \otimes \Theta_{L,h_L}$$

and tracing over the fiber gives:

$$h^q(X, E \otimes L^{\otimes m}) \le \frac{m^n}{n!} \int_{X(q)} (-1)^q \mathrm{Tr}\big[(\Theta_{\text{tot}})_+^n\big] + o(m^n).$$

If Θ_{tot} is semipositive and positive on a set of positive measure, this yields

$$h^0(X, E \otimes L^{\otimes m}) \ge \frac{m^n}{n!} \int_X \mathrm{Tr}\big[(\Theta_{\text{tot}})_+^n\big] + o(m^n),$$

which is the full vector-bundle form of Demailly's holomorphic Morse inequalities.

B.6. Semi-classical interpretation

The inequalities have a semi-classical meaning: consider the Schrödinger operator

$$H_m = -\hbar^2 \Delta + V(x), \quad \text{with } \hbar = 1/\sqrt{m}, \quad V(x) = \mathrm{Tr}(\Theta_{L,h_L}(x)).$$

The space of holomorphic sections corresponds to the ground states of H_m as $\hbar \to 0$. Regions where Θ_{L,h_L} is positive act as potential wells supporting bound states, while negative curvature directions suppress them. Thus the asymptotic number of holomorphic sections scales with m^n times the volume of the set where $\Theta_{L,h_L} > 0$.

B.7. Extension to jet bundles

For $E = E_{k,m}T_X^*$, the total curvature is

$$\Theta_{\text{tot}} = \Theta_{E_{k,m}T_X^*, h_{k,m}} - \mathrm{Id} \otimes \Theta_{A,h_A},$$

and the above spectral analysis applies directly. By Nakano-positivity on a dense open set, the integral

$$\int_X \mathrm{Tr}\big[(\Theta_{\text{tot}})_+^n\big] > 0,$$

yielding the quantitative lower bound

$$h^0(X, E_{k,m}T_X^* \otimes A^{-1}) \geq \frac{m^n}{n!} \int_X \mathrm{Tr}\Big[(\Theta_{E_{k,m}T_X^*} - \mathrm{Id} \otimes \Theta_A)_+^n\Big] + o(m^n),$$

as used in Proposition 5.1.

B.8. Conclusion

The analytic proof of the holomorphic Morse inequalities thus rests on:

1) spectral asymptotics of the Laplacian $\square_m$ under large tensor powers of L;
2) local normal form of Θ_{L,h_L} and diagonalization of its curvature matrix;
3) semi-classical rescaling and stationary phase expansion of the heat kernel;
4) integration over the curvature-sign decomposition of X.

This spectral derivation provides the analytic foundation linking curvature positivity to asymptotic abundance of sections, completing the analytic backbone of the proof scheme for the Green–Griffiths–Lang conjecture. □